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Libro degli abstract

(versione 2)

Contents

Models for wildfire propagation based on a diffusive heat equation with a wind- and topography-driven drift term

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We present a simple model describing wildfire propagation, which takes the start from the original simplified physical fire spread models in [1] and [2], with a modification of the activation term, to avoid problems with a discontinuous term with respect to the unknowns. We discuss some mathematical properties of the model. We propose a simple box-integration scheme for the numerical integration of the model, suitable for numerical simulation in a realistic scenario, with varying velocity field and real topography. We perform some numerical tests to assess the validity of the model.

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Some appearances of distributions in Quantum Mechanics, and related stuff

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We show that some quantum mechanical systems are naturally defined in spaces which are different from $\mathcal{L}^2(\mathbf{R})$. This opens the way to many mathematical questions. In particular, the well known orthogonality of eigenstates of a given self-adjoint operator corresponding to different eigenvalues, suggests to look for a possible extension of the scalar product outside $\mathcal{L}^2(\mathbf{R})$. In this talk we show that these extensions exist in *compatible spaces*, and for suitable sets of tempered distributions. This latter situation is particularly interesting, as we will show.

This talk is based on the papers [1], [2] and [3].

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Trend to equilibrium for the Wigner equation with decoherence

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Decoherence is, roughly speaking, the dynamical loss of the quantum properties of a system. Hence, the study of decoherence is very important for both fundamental physics (e.g., in the analysis of the quantum-to-classical transition) and applied physics (e.g., in quantum computers) [1].

In Ref. [2], starting from the decoherence mechanism analyzed in Ref. [3], the following Wigner equation with decoherence has been introduced:

$$\partial_t w + p\partial_x w = \eta\partial_p(pw) - \nu\,\gamma * w,$$

where η and ν are constants and $\gamma(p)$ is a suitable kernel that depends on the microscopic details of the collisions inducing decoherence. The Wigner-Fokker-Planck equation, a classical model of decoherence [4], corresponds to a quadratic approximation of the Fourier transform of γ.

In this communication we show that, if the ratio ν/η is large enough, then, for $t \to \infty$, the solution w tends to an equilibrium state having a power-law decay. In the Fokker-Planck approximation, the existence of the equilibrium is independent of ν/η and is a Gaussian distribution.

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Modelling Forest Fires Spreading

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Mathematical modelling is an increasingly important tool for managing forest fires, particularly given the enormous quantity of georeferenced data now available. We present a stochastic model continuous in space and time for forest fires spreading [1, 2]. To outline its main features, let us consider a surface Σ on the earth, a point in Σ being denoted by $\mathbf{x} = (x, y) \in \Sigma$. We assume there are positive definite functions $\psi^F(t, \mathbf{x}), \psi^B(t, \mathbf{x}), \psi^G(t, \mathbf{x}), \in L^1(\Sigma, d^2\mathbf{x})$ representing sub-probability densities and describing the state of the forest fire at time t. In particular, $\psi^F(t, \mathbf{x})$ is the sub-probability density for the trees on fire, $\psi^B(t, \mathbf{x})$ the sub-probability density for the burnt trees (firing trees which turned into burnt trees) and $\psi^G(t, \mathbf{x})$ the sub-probability density for the green trees. Their sum satisfies, $\int_\Sigma \left[\psi^F(t, \mathbf{x}) + \psi^B(t, \mathbf{x}) + \psi^G(t, \mathbf{x})\right] d^2\mathbf{x} = 1$. The time evolution of the state is shown to be given by a non-linear system of integro-differential equations for $\psi^F(t, \mathbf{x})$, $\psi^B(t, \mathbf{x})$, $\psi^G(t, \mathbf{x})$,

$$\frac{\partial \psi^F(t, \mathbf{x}_2)}{\partial t} = -\alpha(t, \mathbf{x})\psi^F(t, \mathbf{x}_2) + \psi^G(t, \mathbf{x}_2) \int_\Sigma W(t, \mathbf{x}_2, \mathbf{x}_1)\psi^F(t, \mathbf{x}_1)\, d^2\mathbf{x}_1$$

$$\frac{\partial \psi^G(t, \mathbf{x}_2)}{\partial t} = -\psi^G(t, \mathbf{x}_2) \int_\Sigma W(t, \mathbf{x}_2, \mathbf{x}_1)\psi^F(t, \mathbf{x}_1) d^2\mathbf{x}_1$$

where α is the burning probability rate and W is the transition kernel encoding the fire spreading.

Numerical implementations show forest fire behavior in several frameworks as for example the spreading through a river due to fire spotting.

Real case simulation can be obtained by empirical estimation of the sub-probability density functions (by using, for example, the kernel density estimation method (KDE)) and their time evolution (provided by the model) can be used for probabilistic forecasting: to locate those regions in space where the firing probability is going to increase.

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Quantum Hydrodynamic Models for Charge Transport via Maximum Entropy Closure

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We present a quantum hydrodynamical model for charge transport derived from the Wigner equation using the moment method, where closure is achieved through a quantum version of the Maximum Entropy Principle (QMEP). This approach provides a rigorous framework for handling general energy band structures, going beyond the classical parabolic approximation. Unlike previous models relying on equilibrium Wigner functions, our formulation allows the derivation of explicit closure relations incorporating $\hbar^2$-order quantum corrections via Moyal calculus. We consider the density, energy density, and momentum density as basic hydrodynamic variables, and we construct the associated entropy-maximizing Wigner function under fermionic constraints. The resulting moment system is analyzed in detail, and explicit constitutive relations are obtained for charge transport in both silicon and graphene. The model also enables the derivation of quantum corrections to carrier mobilities under relaxation-time approximations. This work demonstrates how advanced analytical tools from quantum kinetic theory can be leveraged to derive consistent macroscopic transport models with quantum accuracy.

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Bäcklund Transformations & nonlinear soliton equations: scalar versus operator cases

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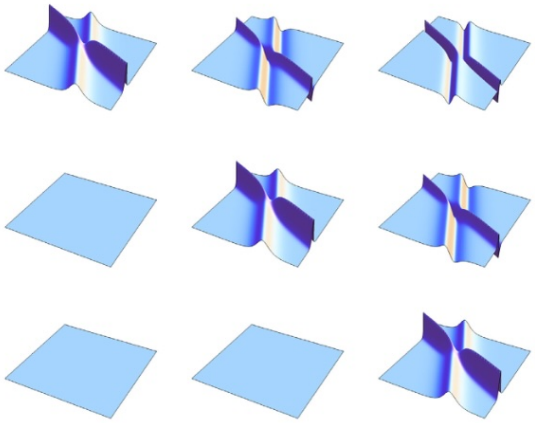
The importance of Bäcklund transformations in investigating non linear differential equations is well known, as testified by many books as, e.g. the classical text by Rogers and Shadwick [1]. In particular, operator valued equations are studied. In this case, the unknown is represented by an operator on a Banach space. Non-Abelian counterparts of non linear evolution equations, such as the Korteweg-deVries equation, can be introduced: they can be connected via Bäcklund transformations. In the case of third order equation, a wide net of links [2], termed *Bäcklund Chart* which exhibits interesting analogies with the the one obtained in the scalar (Abelian) case [3], one is constructed. However, much richer structure is shown in the noncommunicative case. A comparison is provided in [4]. The established connections allow, on one side, to reveal interesting symmetry properties, which can be extended to the full hierarchies since, also in the noncommunicative case, all the equations admit an hereditary recursion operator. On the other side, new solutions can be obtained. In the special case when the unknown operator is finite dimensional, it admits a matrix representation. Such special case is very interesting in applications such as quantum mechanics. Under this perspective, matrix soliton and breather type solutions are obtained [5,6,7]. Matrix solutions of soliton type admitted by the modified KdV equation are studied in [8]: wherein the following figure is included.



The study of a Bäcklund Chart involving 5th order nonlinear equations is in progress [9].

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Wave propagation in nonlocal materials

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We analyze the propagation of thermomechanical waves in materials for which the constitutive equations are allowed to depend on higher-order gradients of the basic unknown quantities. First, we consider the thermoelastic behavior of materials with graded composition. We calculate the velocities of coupled first- and second-sound pulses, propagating through non-equilibrium or equilibrium states. We characterize different types of perturbations, depending on the value of the material coefficients. Starting from the assumption that the deformation of the body can produce changes in its stoichiometry, locally altering the material composition, the possibility of propagation of pure stoichiometric waves is demonstrated. Thermoelastic perturbations generated by the coupling of stoichiometric and thermal effects are analyzed as well [1]. Next, we study the propagation of small-amplitude thermomechanical waves for homogeneous and isotropic thermoelastic bodies, under the validity for the heat flux of a generalized Maxwell-Cattaneo equation that also depends on the deformation, [2]. Finally, we consider a two-component model of superfluidity based on a scalar internal variable whose gradient is the counterflow velocity, i.e., the difference between the velocities of the normal and superfluid components, respectively. It is shown that the first and second sound waves can propagate along the system with velocities depending on the physical parameters of the two fluids. The first sound waves can propagate in the same direction or in the opposite direction to the counterflow velocity, depending on the concentration of the normal and superfluid components [3].

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Remarks on Constitutive Laws for Dry Porous Solids

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In this summary we present some new suggestions concerning the constitutive prescriptions for a model of a complex material, the porous solid, as already introduced in §8.3 of [1] for the incompressible fluid-filled case, which generalizes the theories of voids and take into account remarks [2] on the impossibility of such theories to predict size effects in torsion, as they occur in bone mechanics [3] or rod-shaped specimens of dense polyurethane foams [4]. Surely those remarks stimulate a discussion starting from the more general principles of thermodynamics, even in the context of the entropy principle proposed in [5] and already used in soil mechanics for granular materials [6,7]. Another motivation for our model is the need to correct classical transport equations by involving in the constitutive relations higher–order derivatives of macro- and micro-displacements and/or temperature avoiding classical incompatibilities and recovering a finite speed heat propagation.

We have analyzed the thermomechanics of a complex material, the porous solid, in a scheme in which the ellipsoidal kinematic order parameters describing the microstate, are partially constrained to macrostrain, while the independent variable is directly correlated to the volume fraction of the matrix material and, therefore, is like a measure of the volume variation of the bulk material that results from void compaction or dilation. Pure balance equations are obtained for both macro– and micro–motions by using constraint reactions.

The Cauchy stress tensor results not symmetric, in general, and also depending on acceleration gradients and on a term of Ericksen's type, clearly due to microstructural actions, even in the case of small pores. Moreover, the dissipation functional puts in evidence the effects associated only with void closure and inelastic pore–surface effects. Furthermore, we show that the extra entropy flux, introduced in accordance with Müller's theory, is of reactive origin and balances the extra entropy supply.

Finally, to study the evolution of temperature in a non deforming porous conductor, the microstructural variable are linked to the temperature, obtaining hyperbolic–type evolution of temperature disturbances due to a flux that mimics the interstitial working of [8], a flux due only to microstructural actions.

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Magma Ascent Mechanisms: Conduction, Buoyancy, and Heat Dissipation in the Lithosphere

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We introduce a rather simple model for the ascent of the magma through the lithosphere, based on a mix of conduction, rock melting, and buoyancy, where we take into account the heat dissipation due to the melting process. Moreover a source term is taken into account.

Under suitable assumptions, a hyperbolic linear partial differential equation describing the process is derived from a general thermodynamical model whose constitutive equations are compatible with the second principle of thermodynamics:

$$\tau T_{tt} + T_t - dT_{zz} + uT_z + rT_{tz} - \kappa T = 0; \tag{1}$$

Then, we derive numerically a solution to a physically meaningful initial/boundary condition that can explain many of the experimental observations. More precisely, we suggest a mechanism for the magma emplacement able to explain the different depths of emplacement depending on one of the model parameters. Moreover, this parameter can be tuned to obtain the disappearance of the emplacement, which justify the occurrence of hot spots.

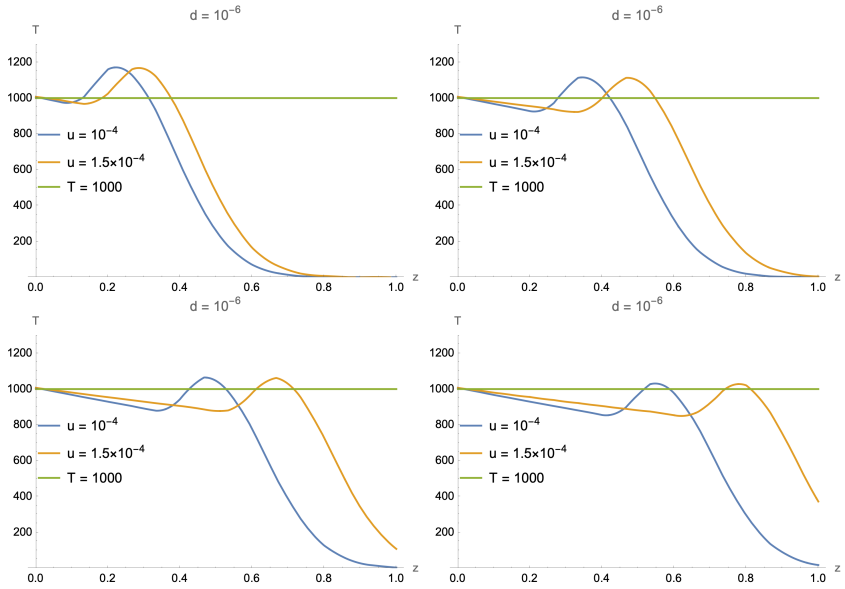

Figure 1: Profiles of the numerical solution at different times for $d = 10^{-6}$.

A thermodynamical suspension model for blood

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A complete thermodynamical analysis for a blood model, based on mixture theory, is performed [1]. The model is developed considering the blood as a suspension of red blood cells (solid component) in the plasma (fluid component), and taking into account the temperature effects. Furthermore, two independent scalar internal variables are introduced accounting for additional dissipative effects. Using Clausius-Duhem inequality, the general thermodynamic restrictions and residual dissipation inequality are derived. The thermodynamic admissibility with the second law of thermodynamics is assessed by means of the extended Coleman-Noll procedure [2]; in one space dimension we exhibit a solution of all the thermodynamical constraints.

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Transient waves in linear dispersive media with dissipation: an approach based on the steepest descent path

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For uni-axial waves in a semi-infinite medium ($x \geq 0$), initially quiescent and denoting by r the "response" of the system chosen among the field variables, the pulse $r(x,t)$ can easily be expressed at any time $t \geq 0$ and position $x \geq 0$ by inverting a Laplace transform. In fact we get, for media admitting a finite wave-front velocity c and a complex index of refraction $n(s)$

$$r(x,t) = \frac{1}{2\pi i} \int_{Br} ds\, \widetilde{r}_0(s)\, \mathrm{e}^{\mathrm{s[t-n(s)x/c]}}\,, \tag{1}$$

in which s is the complex Laplace parameter, Br denotes the Bromwich path and the function $\widetilde{r}_0(s)$ is the Laplace transform of the source pulse at $x = 0$ for $t \geq 0$, i. e. $r(0,t) = r_0(t)$. We recall that the Bromwich path is any infinite vertical line lying in the s-complex plane to the right of all singularities of $\widetilde{r}_0(s)$ and $n(s)$.

In order to decrease the numerical difficulties for the Laplace inversion we propose to deform the original path of integration (the Bromwich path) into an equivalent one (unless possible contributions of singularities) that turns out to be more convenient. Such path is the steepest descent path through the saddle points of the complex function

$$F_\mu(s) = s[1 - \mu\, n(s)]\,, \quad \mu = \frac{x}{ct}\ (0 \leq \mu \leq 1), \tag{2}$$

i.e. the path along witch the imaginary part of the phase function F_μ is constant. We have the possibility to check the validity of our proposed method by comparing our numerical results with the analytic solutions available in two particular cases of a signaling problem with complex index of refraction $n(s)$ provided by the Klein-Gordon equation, that is a relevant model equation that takes into account both dispersion and dissipation. This equation denoted by KGD reads

$$r_{tt} + a\, r_t + b\, r = c^2\, r_{xx}\,, \quad r = r(x,t)\,, \tag{3}$$

where the subscripts denote partial derivatives and a, b, c are non negative constants characteristic of the wave phenomenon under consideration, including elastic and electromagnetic waves in dispersive and dissipative media.

The complex index of refraction associated to the KGD equation is $n(s) = [1 + a/s + b/s^2]^{1/2}$, and exhibits in the complex s plane two branch points that are complex conjugate (c.c) for $\Delta = b - a^2/4 > 0$ and real for $\Delta = b - a^2/4 < 0$. The steepest descent paths, which turn out to be qualitatively different for these two cases, have been obtained and the numerical integration provides results which are in excellent agreement with the analytical solutions.

As a basic reference we may suggest the book by Mainardi [1] that contain many useful references,

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On the Navier-Stokes equations: a possible gap related to the energy equality of a weak solution

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It is well known that a weak solution *a priori* enjoys an energy inequality. Hence, we investigate on the possible gap:

$$\|v(s)\|_2^2 - 2\int_s^t \|\nabla v(\tau)\|_2^2 d\tau - \|v(t)\|_2^2 \geq 0\,. \tag{1}$$

We look for the existence of a Leray's weak solution enjoying the energy equality. We are unable to fully prove the result. However, we show that if there exists a possible gap for the energy equality, i.e. at (1) > 0, then the gap is represented by means of a suitable additional dissipation. The additional dissipation is given in terms of the "kinetic energy".

We are not able to detect a turbulence character for these quantities, or, more in general, to understand the physical meaning of the anomalous energy dissipation.

The gap vanishes in the case of a further "small regularity" of the weak solution.

This result is an existence result, we do not achieve the result for all possible the Leray-Hopf weak solutions.

Thermodynamic and symmetry requirements for electromagnetic solids

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The talk revisits the subject of constitutive equations in electromagnetism on the basis of the symmetry requirements arising from the balance of angular momentum. First, the balance of angular momentum is considered, thus obtaining a symmetry condition that is applied as a mathematical constraint on admissible constitutive equations. Next, thermodynamic restrictions are also investigated and a further symmetry condition is established. The joint validity of the two symmetry conditions implies that the dependence on electromagnetic fields has to be through variables involving deformation gradients. These variables constitute two classes that prove to be Euclidean invariants. The simplest selection of the variables is just that of the Lagrangian fields of the literature. Furthermore, the two classes of variables yield different types of magnetostriction (i.e. positive and negative). The generalization to dissipative and heat-conducting solids is obtained by using the entropy production as a constitutive function.

Entropy principle for a nanoscale thermal model with no-slip boundary conditions

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A complete thermodynamical analysis for a nanoscale thermal model is carried out [1]. The model is developed considering the heat flux decomposed into two contributions: bulk and wall [2]. Furthermore, to account the effects of phonon-wall interactions, second-order no-slip boundary conditions are introduced [3,4,5,6,7,8]. Based on the second law of thermodynamics, general thermodynamic restrictions and residual dissipation inequality are derived. The underlying idea is to consider the heat flux at the boundary, i.e the wall heat flux, as an independent variable. This allows for the derivation of general boundary conditions from the second law requirement that the rate of entropy production must be non-negative. In parallel, evolution equations for the bulk heat flux are also formulated.

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On the validity of the Fourier law in a Resonant Tunneling Diode

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Semiconductor devices that rely on quantum tunneling through potential barriers are playing an increasingly important role in advanced microelectronic applications, including multiple-state logic and memory devices and high frequency oscillators. Monte Carlo simulations provide detailed information on carrier transport within such devices, but the computational burden limits its use for many device engineering applications. To treat the nonlocal transport effects that occur in small devices, macroscopic transport equations promise acceptable accuracy with a much reduced computational burden. Macroscopic equations are derived by taking moments of the Wigner-Boltzmann transport equation (BWTE), but numerous simplifying assumptions are needed to formulate a tractable set of equations. As a result of these simplifying assumptions, nonphysical numerical results can occur. During these years a "smooth" quantum hydrodynamic (QHD) model for semiconductor devices, derived by a Chapman-Enskog expansion of the Wigner-Boltzmann equation, has been proposed [1]. With this expansion a generalized Fourier law for the heat flux has been obtained, including both a classical and a quantum term and incorporates important effects of the higher moments of the Wigner-Boltzmann equation that are omitted in the fluid dynamical approximation. The common approximation of the heat flux by Fourier's law is shown to differ substantially from the actual heat flux [2].

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Modeling the spread of Xylella fastidiosa: mathematical insights into a contested plant epidemic

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Outbreaks of Xylella fastidiosa, a plant-pathogenic bacterium linked to the decline of olive trees in southern Europe, have drawn widespread attention due not only to their ecological and economic consequences but also to the intense debate surrounding their causes and management. In this talk, we present a mathematical and numerical analysis [1] of an eco-epidemiological model designed to capture the complex dynamics of Xylella fastidiosa spread through plant and insect populations .

The model incorporates key ecological interactions and transmission mechanisms [2], and we rigorously prove the existence and uniqueness of solutions under biologically realistic conditions [1]. Numerical simulations are then used to explore various outbreak scenarios, providing insight into how the disease might evolve under different environmental and intervention parameters.

By offering a clearer picture of the infection dynamics, this work contributes to the scientific understanding needed to assess control strategies—especially in a context where narratives around the disease have often been shaped by misinformation, oversimplification, or political agendas. Our results highlight the importance of evidence-based modeling in distinguishing biological reality from speculation, particularly when irreversible actions such as mass tree removal are at stake.

Figure 1: The vector *Philaenus spumarius*. Source: http://www.entomart.be/html3/INS-6780.html

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Optimal control and uncertainty quantification of the semiclassical Boltzmann equation for charge transport in graphene layers and ribbons

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Understanding and predicting charge transport in graphene is essential for the design of next-generation nanoelectronic and quantum devices. Due to graphene's unique electronic properties, such as linear dispersion relation and high carrier mobility, accurate modeling of transport phenomena requires frameworks that go beyond classical drift-diffusion. In this presentation, we explore recent advancements in applying the semiclassical Boltzmann transport equation to graphene-based systems.

First, we focus on optimal control of transport behavior through an applied electrical field. This control enables the manipulation of average momentum and current density in graphene. In our work [1], these control problems are formulated using an ensemble cost functional with quadratic H^1 cost. Well-posedness of the control problem is proved, and the characterization of optimal controls by an optimality system is discussed. This system is approximated by a discontinuous Galerkin scheme and solved using a nonlinear conjugate gradient method.

Next, we consider the challenge of uncertainty quantification (UQ), arising from incomplete or imprecise knowledge of key material and device parameters, e.g. the effective band gap or the strength of applied electric field. In our work [2], we investigate the uncertainty quantification in graphene nanoribbons via a particle Galerkin approach. To this end, we develop an efficient particle scheme which allows us to parallelize the computation and then, after a suitable generalization of the scheme to the case of random inputs, we present a Galerkin reformulation of the particle dynamics, obtained by means of a generalized Polynomial Chaos approach, which allows the reconstruction of the kinetic distribution.

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A revisitation of the principle of material objectivity: a Lie symmetry approach

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In continuum mechanics and thermodynamics, the principle of material frame-indifference, also referred to as the principle of material objectivity, is a fundamental postulate introduced by Truesdell and Noll in 1965 [1]. It represents the main requirement for assigning constitutive relations and identifying objective quantities.

Roughly speaking, the principle states that the constitutive relations must be invariant with respect to a change of the observer, namely by a transformation like

$$\mathbf{x}^\star = Q(t)\mathbf{x} + \mathbf{c}(t), \qquad t^\star = t + t_0$$

where Q is an arbitrary time dependent orthogonal matrix, and $\mathbf{c}$ an arbitrary time dependent vector. This principle implies, the invariance of scalar functions, and suitable transformations for vectors and tensors, say

$$f(\mathbf{x}^\star) = f(\mathbf{x}), \qquad \mathbf{v}(\mathbf{x}^\star) = Q\mathbf{v}(\mathbf{x}), \qquad T(\mathbf{x}^\star) = Q^T T(\mathbf{x}) Q.$$

A long series of strong disputes accompanied this principle; a historical view (at least, until 2008) can be found in [2].

We exploit the transformations for scalar, vectorial and tensorial functions in order a *principle of material frame indifference* is fulfilled by using the infinitesimal invariance with respect to the so called fundamental group of mechanics [3], *i.e.*, the ten–parameter Lie goup made of isometries, time translation and Galilean transformations. So doing we can provide a suitable framework for identifying objective quantities, providing the desired transformation rules for scalars, vectors and tensors, and giving the representation for isotropic constitutive functions. This approach provides the same results obtainable by the Truesdell and Noll principle in most of the cases, but not all. The differences are discussed.

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The motion of a rigid body in a viscous fluid: new results for strong solutions, uniqueness and integrability properties.

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The motion of a rigid body $\mathcal{B}$ in a viscous fluid $\mathcal{F}$ has been the subject of intensive research over the last few decades. We do not make any assumption about the shape of the body, we only assume that the boundary $\partial\mathcal{B}$ is sufficiently smooth. Concerning the fluid, we assume that it fills the entire space Ω exterior to $\mathcal{B}$. Hence, we have $\Omega = \mathbb{R}^3 \setminus \mathcal{B}$. The equations that describe the motion of the system $\mathbf{S} := \{\mathcal{F}, \mathcal{B}\}$, in a frame attached to $\mathcal{B}$ with the origin in its center of mass, read as follows:

$$\begin{cases} u_t - \Delta u = -[(u - V) \cdot \nabla u + \omega \times u] - \nabla \pi \quad \forall (t,x) \in (0,T) \times \Omega\,, \\ \nabla \cdot u = 0 \quad \forall (t,x) \in (0,T) \times \Omega\,, \\ u(t,x) = V(t,x) = \xi(t) + \omega(t) \times x \quad \forall (t,x) \in (0,T) \times \partial\Omega\,, \\ \lim_{|x| \to \infty} u(t,x) = 0 \quad \forall t \in (0,T)\,, \\ \dot{\xi} + \omega \times \xi + \int_{\partial\Omega} \mathbb{T}(u,\pi) \cdot n = 0 \quad \forall t \in (0,T)\,, \\ I \cdot \dot{\omega} + \omega \times (I \cdot \omega) + \int_{\partial\Omega} x \times \mathbb{T}(u,\pi) \cdot n = 0 \quad \forall t \in (0,T)\,, \\ \xi(0) = \xi_0\,, \quad \omega(0) = \omega_0\,, \\ u(0,x) = u_0(x)\,, \quad \forall x \in \Omega\,. \end{cases} \tag{1}$$

The previous model has been exhaustively described in [1].

In this presentation, we introduce two novel results concerning strong solutions to the system describing the motion of a viscous fluid in interaction with a rigid body.

The framework of solutions considered corresponds to that introduced by Galdi and Silvestre [2].

Firstly, under an integrability condition on the gradient of the initial data, we demonstrate that the strong solution constructed by Galdi and Silvestre possesses the additional regularity property $u_t \in L^2(0,T;L^2(\Omega))$.

Secondly, we establish a uniqueness result: if a strong solution satisfies a further integrability condition, then it is unique. Moreover, this additional integrability requirement is automatically fulfilled when the gradient of the initial data meets the assumption that ensures the aforementioned regularity of the time derivative of the velocity field.

(Joint work with P. Maremonti)

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Analytical approach for two dimensional Cattaneo equation with non-homogeneous boundary conditions

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The classical Fourier law is no longer applicable when studying thermal transient problems at the nanoscale or when describing heat transfer phenomena in heterogeneous materials. Non-Fourier models are of great interest in the engineering field, particularly due to the use of heat sources in many applications. In this paper, we provide an analitycal solution to the Maxwell-Cattaneo heat equation with homogeneous initial conditions and non-homogeneous time-dependent boundary conditions, restricting ourselves to the linear regime and a two-dimensional situation. An analytical solution of the heat wave-type equation is derived after substituting the time-dependent heating function acting on the boundary with a time- and space-dependent heat source. An exact solution is obtained using the method of superposition, combining the homogeneous transient case and the inhomogeneous steady state.

Finally, the time evolution of the temperature history and the heat flux profile with time is represented for both the original problem and the one approximated with a space- and time-dependent heat source, thus validating the solution method.

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Ballistic electron transport described by a fourth order Schrödinger equation

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A fourth order Schrödinger equation for the description of charge transport in semiconductors in the ballistic regime is proposed with the inclusion of non parabolic effects in the dispersion relation in order to go beyond the simple effective mass approximation. Similarly to the standard (second order) Schrödinger equation, the problem is reduced to a finite spatial domain with appropriate transparent boundary conditions to simulate charge transport in a quantum coupler [1, 2, 3], where an active region representing an electron device is coupled to leads which take the role of reservoirs. Some analytical properties are investigated and a generalized formula for the current is obtained. Numerical results show the main features of the solutions of the new model. In particular, an effect of interference appears due to a richer wave structure than that arising for the second order Schrödinger equation in the effective mass approximation. The results are based on the paper [4].

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Mathematical Modeling and Analysis of Semiconductor Laser Diodes

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This talk explores advanced mathematical models for semiconductor laser diodes, built upon a system of coupled drift-diffusion equations and the Helmholtz equation. We will examine the theoretical framework required to ensure well-posedness, specifically focusing on the conditions that guarantee existence and uniqueness of solutions. Special attention will be given to the optical power output, described through the photon balance relation, and its dependence on the underlying electrostatic and chemical potentials within the semiconductor structure.

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Fractional differential equations for seismic wave propagation

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Recent literature is reach in models based on fractional differential equations able to describe nonlinear phenomena in the field of engineering, physics as well as natural sciences, attempting the fitting with real phenomena, such as in describing solid Earth dynamics, where fractional calculus can be added to account for spatial and temporal memories. The main goal of is to define the large scale Earth surface dynamics by means of time and space fractional model, that generalize the Initial Boundary Value Problem (IBVP) introduced in [1, 2]. Both analytical and numerical approaches are proposed and a general procedure for solving linear and non-linear fractional differential equations describing the interactions between seismic wave field and near-surface geological structures is presented. In details, in order to explore the exact solution of the FPDE, a recent procedure which combines an extension of the classic Lie symmetry theory to FPDEs with the numerical methods [3], [4]has been applied. Otherwise, an asymptotic approach with respect to a small parameter linked to the fractional order derivative, has been adopted, [5]. From a numerical point of view, the main idea is to integrate the IBVP through an original numerical procedure that merge advanced numerical techniques [6] with multistep–predictor corrector method for solving differential equations of fractional order.

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Density matrices and entropy operator for non-Hermitian quantum mechanics

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In functional analysis, the analysis of Hilbert spaces and of the operators acting on them is quite important, one of these operators is the so-called density matrix (DM), whose role turns out to be particularly useful for open quantum systems [1].

In the *standard* literature on quantum mechanics, a DM ρ_0 is, first of all, a self-adjoint bounded operator: $\rho_0 = \rho_0^\dagger$, this implies that ρ_0 admits a set of eigenvectors which, under suitable assumptions, form an orthonormal basis of the Hilbert space where ρ_0 acts. But in the past few decades it becames clearer and clearer that ONB are not always the most natural set of vectors appearing when loosing self-adjointness. In many cases, one has to consider bi-orthogonal sets of vectors, which could be Riesz bases or not [2], therefore, we aim to investigate the implications for DMs [3].

Here we start a detailed analysis of what a DM can be thought to be for a quantum mechanical system driven by a non self-adjoint Hamiltonian. In particular, we will consider two different situations: in the first (and easiest) one, the new DM is simply similar to ρ_0 i.e:

$$\rho = R\rho_0 R^{-1}, \tag{1}$$

and the similarity is implemented by a bounded non unitary operator R with bounded inverse. In this case, as one can easily imagine, bi-orthogonal Riesz bases will be relevant. However, we will also consider the case in which the *new* DM ρ is not similar to ρ_0, but still ρ and ρ_0 are linked by a intertwining operator, i.e.:

$$\rho R = R\rho_0, \tag{2}$$

We will see that, in this case, the situation is much more delicate, but still interesting.

We should also stress that the role of DMs is relevant also in connection with quantum mechanical states, pure or not, and with the definition of an entropy operator. These aspects will also be considered in our analysis, for our *extended* DMs.

Finally, we try to extend also the concept of *fidelity* between two different states [6], indivituated by the relative extended density matrices.

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Euler and Navier-Stokes equations with data analytic close to the boundary

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This work investigates the analyticity of solutions to the Euler and Navier-Stokes equations. While previous studies [1, 3, 4] have generally assumed that the initial data are analytic throughout the entire domain, the present study shows that the Euler equations admit solutions with analytic initial data localized near the boundary, in both two and three dimensions. The analysis is carried out using the Leray formulation of the equations. By constructing solutions that are analytic only in a neighborhood of the boundary, this work sheds light on the minimal regularity assumptions required to explore the vanishing viscosity limit of the Navier-Stokes equations. In the concluding part of the talk, we will discuss how this approach may be extended to the Navier-Stokes equations. In two dimen- sions, the problem with data analytic near the boundary has recently been resolved in [2]. Our goal would be to establish the corresponding result in three dimensions, including an existence theorem for the Navier-Stokes equations that holds uniformly in time with respect to the viscosity.

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The one-fluid extended model of superfluid helium II: recent results

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The aim of the talk is to show a depeer comparison between the two main existing models of superfluid helium: the two-fluid model, proposed by Landau in 1941 [1, 2], and the one-fluid extended model [3, 4], proposed by using Extended Thermodynamics.

In particular, we perform some numerical experiments of these models for a direct comparison with the experiments by Guo's group [5, 6] in heat transport in superfluid helium. The numerical experiments will regard the profiles of the so-called normal and superfluid components in 2D counterflow turbulence for the two-fluid model, and the heat flux and the main velocity for the one-fluid extended model. To make progress, we also perform numerical simulations where we allow a slip velocity of the viscous component at the walls, and observe how this impacts on velocity fields and density profiles of distribution of quantized vortices.

Acknowledgements: The results are partially supported by the PNRR project ECS00000022 "SiciliAn MicronanOTecH Research And Innovation CEnter (SAMOTHRACE)" and the PRIN project 2022TMW2PY 002 "Transport phonema in low dimensional structures: models, simulations and theoretical aspects"

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On Well Posedness Results for the 2D-Viscous MHD Equations

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We consider the large time behaviour of the solutions tp the viscous magneto-hydrodynamic (MHD) equations in the whole space $\mathbb{R}^2$. The two dimensional MHD equations describe the evolution of an incompressible electrically conducting fluid moving through a magnetic vector field. The interaction between the fluid velocity and the magnetic field is described by the coupling between the Navier-Stokes equations and the Maxwell's equations. We consider vortex-current patchs as initial data.

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HOW TO TEST NONEQUILIBRIUM THERMODYNAMICS?

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There are many different approaches to non-equilibrium thermodynamics (extended, rational, classical, GENERIC, etc.) the other fields, where thermodynamics plays an important role are even more numerous. It is then argued that all such attempts of a general theory face the challenge of validation and must pay attention to the following means and measures of performance and validity:

1. Prediction of new experiments.
2. Resolution of existing paradoxes.
3. Theoretical compatibility. For thermodynamics the explanation of universality is distinguished.

These measures of validity may differ for different theoretical approaches, but one aspect deserves special attention: any proposal must demonstrate compatibility with classical theories: classical thermodynamics and continuum physics, including heat conduction, continuum mechanics and electrodynamics. A consistent connection to the classical background can validate the theory with an enormous amount of empirical facts (observations, experiments and engineering experience), and it is also a source of paradoxes, possible new experiments that can be solved, further increasing the credibility of any theory.

The other standard test method is the experimental validation. Then one must derive new prediction, and design experiments that can test them. In the presentation some fundamental predictions of nonequilibrium thermodynamics are discussed and some possibilities and ongoing projects of testing them.

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Deformable ferroelectrics models: a thermodynamically consistent approach

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Within the framework of continuum thermodynamics, we propose a phenomenological, non-isothermal, vector-valued model of the ferroelectric phase transition based on a thermodynamic approach. The scheme adopted here applies to piezo-ferroelectric materials undergoing large deformations [1-3].

The modeling of the constitutive properties are given by using referential, Euclidean invariant quantities and the ferroelectric polarization vector is decomposed into the sum of a reversible (piezoelectric) component, and a residual (remnant) part which is considered as an independent variable.

The novelty of our approach is based on constitutive functions that depend on a set of variables, including the residual polarization gradient, as well as their time derivatives. This (weakly) nonlocal assumption is compatible with continuum thermodynamics provided that the second law is formulated in a nonlocal form, where the entropy production is represented as the sum of a non-negative supply and a flow term due to an extra-flux vector. Both contributions take into account the exchange of entropy at the interface between ferroelectric domains and are assigned by means of constitutive functions.

We obtain a completely original result regarding the explicit evolution equation of the residual polarization vector, both for high and low temperatures, when large deformations are involved. In particular, the model is able to describe the evolution of both the intensity and the direction of the residual polarization.

The driving term of the evolutionary system is the variation of the Gibbs free energy functional. Hence, within the Ginzburg-Landau-Devonshire setting, some appropriate expressions of the free energy are proposed. For both isotropic and anisotropic materials the free energy due to polarization is constructed and the nature of the relative maxima and minima is discussed.

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